22. To build up a population of trout in a small lake, 200 young trout are added each year. In addition, the population increases its own numbers by 20% each year. Let x_n denote the size of the population after n years.
(a) If $x_0 = 1200$, determine when $x_n \geq 2800$.
(b) Once $x_n \geq 2800$, the lake is no longer stocked and fishers will catch 600 fish per year. What is the fate of the population?

The change in the population size is 20% times the population size and then plus 200 trout, which can be written as

$$\Delta x = 20\% \cdot x_n + 200$$

$$x_{n+1} - x_n = 0.2 \cdot x_n + 200$$

$$x_{n+1} = x_n + 0.2 \cdot x_n + 200$$

$$x_{n+1} = 1.2 \cdot x_n + 200$$

which is a linear first order difference equation where $a = 1.2$ and $b = 200$. So the equilibrium for this equation is $A = \dfrac{b}{1-a} = \dfrac{200}{1-1.2} = -1000$.

For part (a), if $x_0 = 1200$, then the solution to the difference equation is

$$x_n = (x_0 - A) \cdot a^n + A$$

$$x_n = (1200 - (-1000)) \cdot (1.2)^n + (-1000)$$

$$x_n = 2200 \cdot (1.2)^n - 1000$$

and we want to find when $x_n \geq 2800$.

$$2800 = 2200 \cdot (1.2)^n - 1000$$

$$3800 = 2200 \cdot (1.2)^n$$

$$\tfrac{19}{11} = (1.2)^n$$

$$\ln\left(\tfrac{19}{11}\right) = \ln(1.2)^n$$

$$n \cdot \ln(1.2) = \ln\left(\tfrac{19}{11}\right)$$

$$n = \frac{\ln\left(\frac{19}{11}\right)}{\ln(1.2)} \approx 3$$

You can check the difference equation using $x_0 = 1200$ to see that $x_3 = 2801.6$.

Part (b) is easier to do if we start with a new difference equation

$$x_{n+1} = 1.2 \cdot x_n - 600 \quad \text{where } x_0 = 2800$$

This has an equilibrium of $A = \dfrac{-600}{1-1.2} = 3000$.

Since 1.2 is greater than 1, this equilibrium is unstable, which means that if we start with $x_0 = 2800$, which is less than 3000, the population will decline over time until it eventually goes extinct.

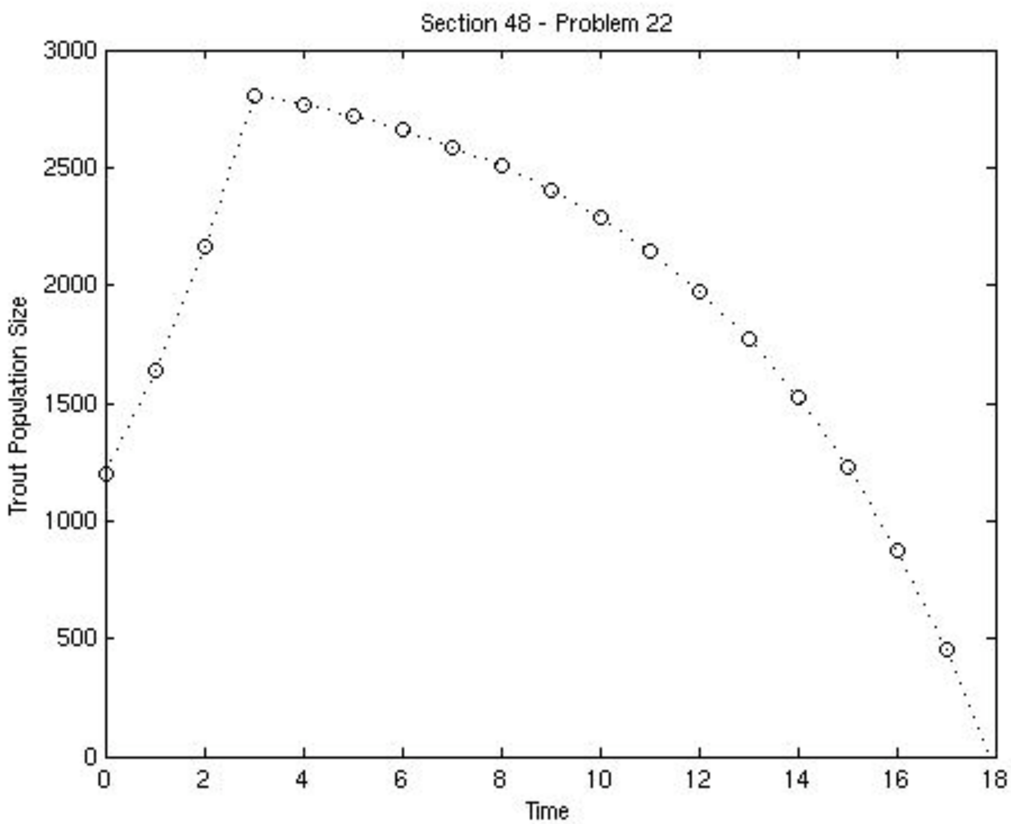
So how much fishing could we allow so that the trout population won't go extinct? Essentially, this means we want the equilibrium to be less than or equal to the initial value (in this case, 2800) so that the population will grow (or at least not decline).

$$A \le 2800 = \frac{-h}{1-1.2}$$

$$2800 = \frac{-h}{-0.2} \qquad \text{where } h \text{ is the number of trout that are caught}$$

$$h = 2800 \cdot 0.2 = 560$$

So the most amount of fishing we should allow is 560. If 560 fish are caught each year, the population size will remain at 2800. If less than 560 fish are caught each year, the population size will grow without limit. If more than 560 fish are caught each year, the population size will decline until it goes extinct. Notice that all this depends on the initial value being 2800. If there were a different number of fish in the pond when we starting allowing fishing, we would have to recalculate the maximum amount of fishing allowed.



24. A population of buffalo can increase its numbers by about 15% each year. Let x_n be the population count after n years, and assume that h buffalo are removed from the herd at the end of each year.
(a) Find x_n in terms of h if $x_0 = 1000$.
(b) Find the largest h so that $x_{10} \geq 2000$.

The change in the population size is 15% times the population size and then minus h buffalo, which can be written as

$$\Delta x = 15\% \cdot x_n - h$$

$$x_{n+1} - x_n = 0.15 \cdot x_n - h$$

$$x_{n+1} = x_n + 0.15 \cdot x_n - h$$

$$x_{n+1} = 1.15 \cdot x_n - h$$

which is a linear first order difference equation where $a = 1.15$ and $b = -h$. So the equilibrium for this equation is $A = \frac{b}{1-a} = \frac{-h}{1-1.15} = \frac{20}{3}h$.

Since 1.15 is greater than 1, this equilibrium is unstable, which means that if we start with an initial population size, x_0, that is less than the equilibrium, $\frac{20}{3}h$, the population will decline over time until it eventually goes extinct.
So the maximum amount of harvesting we should allow is $h = \frac{3}{20}x_0$.

For part (a), if $x_0 = 1000$, then the solution to the difference equation is

$$x_n = (x_0 - A) \cdot a^n + A$$

$$x_n = \left(1000 - \tfrac{20}{3}h\right) \cdot (1.15)^n + \tfrac{20}{3}h$$

and the maximum amount of harvesting we should allow is $h = \frac{3}{20} \cdot 1000 = 150$.

For part (b), if we want $x_{10} \geq 2000$, then to find the largest h, we solve

$$2000 = \left(1000 - \tfrac{20}{3}h\right) \cdot (1.15)^{10} + \tfrac{20}{3}h$$

for h.

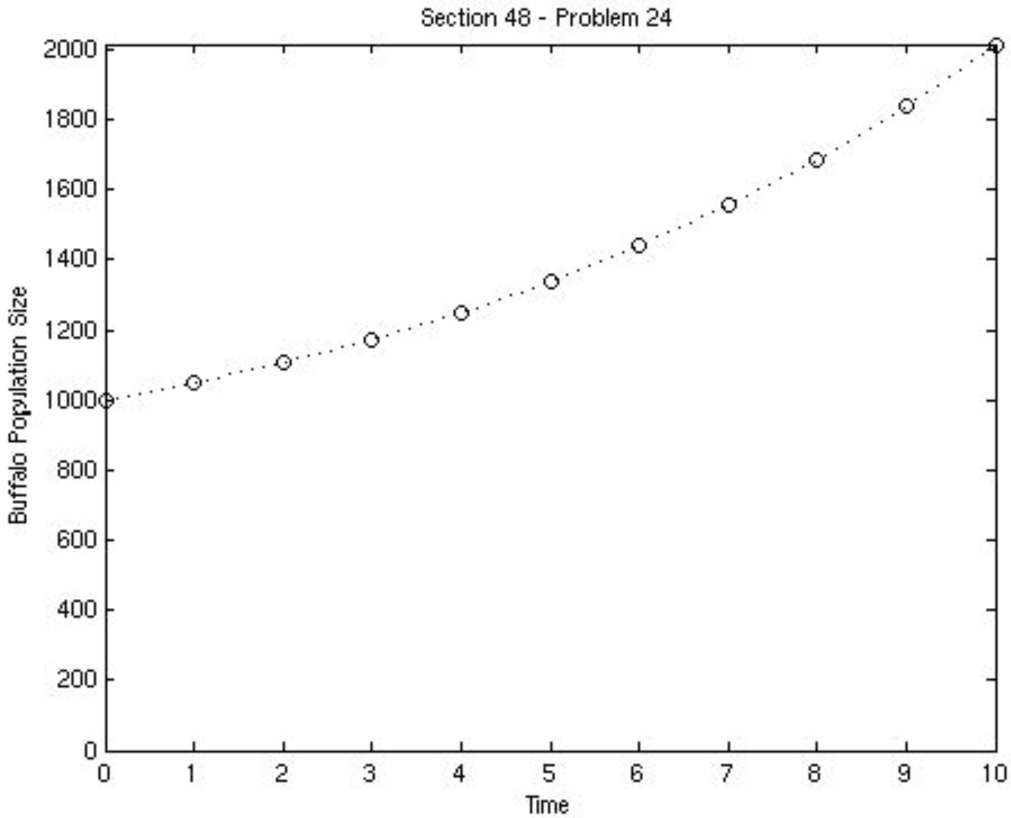
$$2000 = (1000 - 6.6667h) \cdot 4.04556 + 6.6667h$$

$$2000 = 4045.56 - 26.97h + 6.6667h$$

$$-2045.56 = -20.3033h$$

$$h \approx 100$$

You can check the difference equation using $x_0 = 1000$ to see that $x_{10} = 2015$ when $h = 100$ and $x_{10} = 1995$ when $h = 101$.


Section 48 - Problem 24
Buffalo Population Size
0
200
400
600
800
1000
1200
1400
1600
1800
2000
0
1
2
3
4
5
6
7
8
9
10
Time

25. A population of pheasants lives on a small island. Because of limited resources, the island can sustain no more than K individuals. The population growth during the year, $\Delta x = x_{n+1} - x_n$, can be expected to decrease as x_n approaches K. Assuming that Δx is directly proportional to $K - x_n$, find x_n given that $x_0 < K$. What is $\lim_{n \to \infty} x_n$?

Since the change in the population size is directly proportional to $K - x_n$, this can be written as

$$\Delta x = \gamma \cdot (K - x_n)$$
$$x_{n+1} - x_n = \gamma \cdot K - \gamma \cdot x_n \qquad \text{(where } \gamma \text{ is some constant greater than zero)}$$
$$x_{n+1} = x_n - \gamma \cdot x_n + \gamma \cdot K$$
$$x_{n+1} = (1-\gamma) \cdot x_n + \gamma \cdot K$$

which is a linear first order difference equation where $a = 1 - \gamma$ and $b = \gamma \cdot K$. So the equilibrium for this equation is $A = \dfrac{b}{1-a} = \dfrac{\gamma \cdot K}{1-(1-\gamma)} = K$.

Since $\gamma > 2$ corresponds to $a < -1$, the equilibrium is only stable when $0 < \gamma < 2$. Also, since $1 < \gamma < 2$ corresponds to $-1 < a < 0$, the population size will "oscillate" to K when $1 < \gamma < 2$.

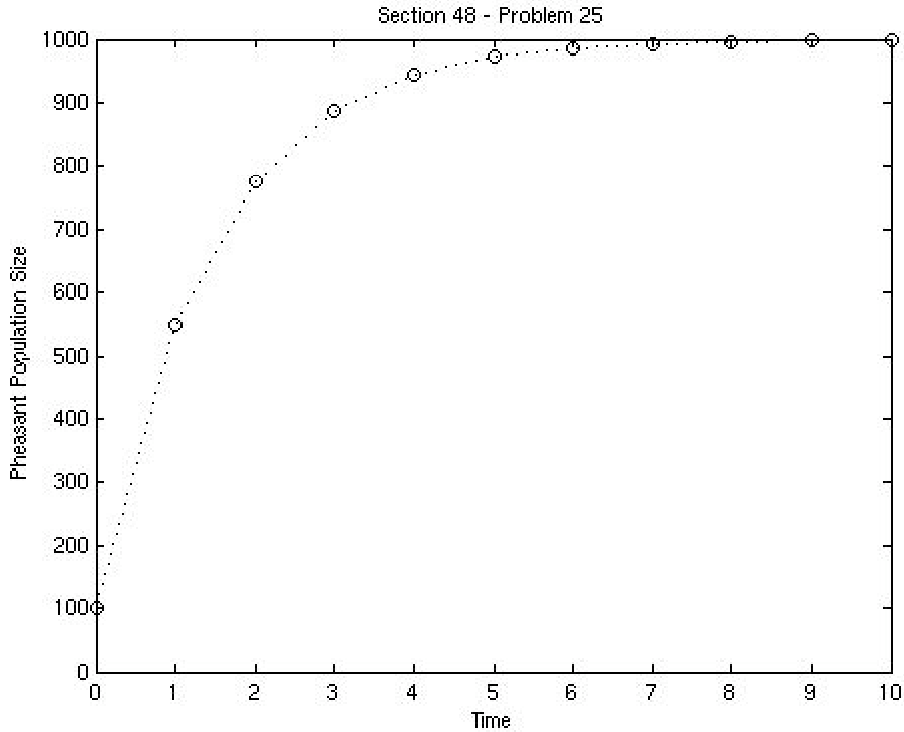
The solution to the difference equation is

$$x_n = (x_0 - K) \cdot (1-\gamma)^n + K$$

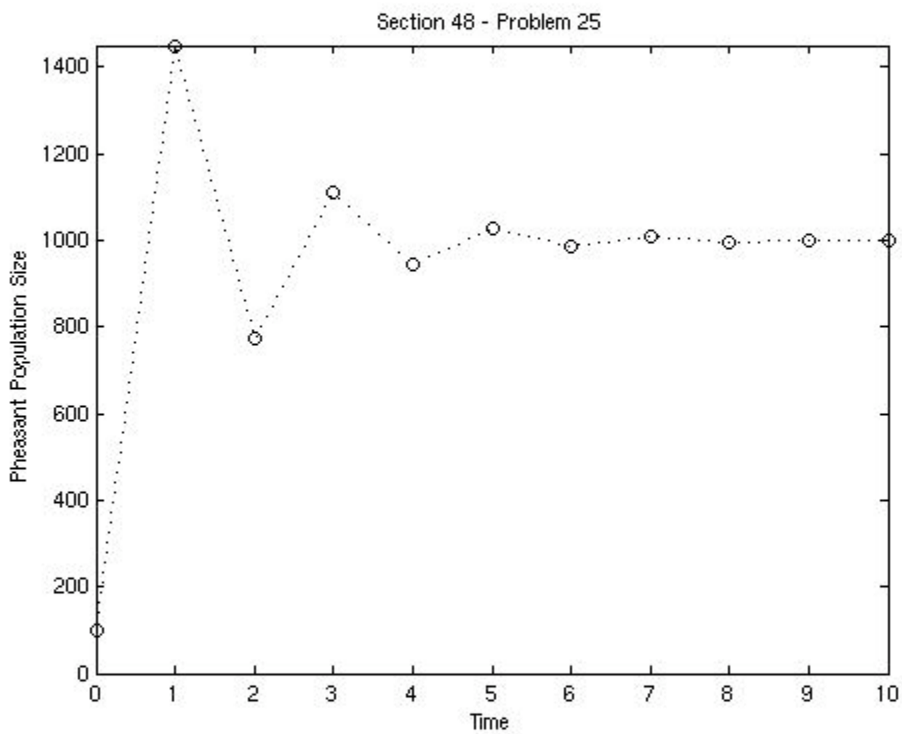
and

$$\lim_{n \to \infty} x_n = K \qquad \text{as long as } 0 < \gamma < 2.$$

$x_0 = 100,\ K = 1000,\ \gamma = 0.5$



$x_0 = 100,\ K = 1000,\ \gamma = 1.5$



27. The body eliminates 10% of a certain drug each hour. Suppose that doses of 200 milligrams are given every 6 hours.
(a) Find the maximum drug level in the body.
(b) Find the amount of drug in the body 24 hours after the first dose.
(c) How frequently should the drug be administered to build up a maximum drug level of 1000 milligrams?

This problem must be broken up into (at least) two parts. First, we need to handle the elimination of the drug from the body, which is a continuous time process. Then, we can combine it with the drug dosage, which is a discrete time process.

Experimental evidence tells us that for most drugs, the amount of the drug in the body decays exponentially over time. (See example 48.5 on pages 537-538 of the textbook.)

Since the dose is being administered every 6 hours, and we are only told how much the drug decays every hour, we need to calculate how much of the drug will remain after 6 hours. We can calculate this using the exponential decay equation

$$N(t) = N_0 \cdot R^t$$

where $R = 0.9$ (since if 10% of the drug decay, then 90% is left) and t is measured in hours.

For simplicity, we'll assume that $N_0 = 200$, which is the amount of each dose. Then, the amount of the drug remaining in the body after 6 hours is

$$N(6) = 200 \cdot (0.9)^6 = 200 \cdot 0.531441 = 106.2882$$

which means the proportion of the drug remaining in the body after 6 hours is

$$\frac{106.2882}{200} = 0.531441 \approx 53\%$$

or, in other words, the body eliminates 47% of the drug every 6 hours (that is, between doses).

Now, if we include the drug dosage, the change in the amount of the drug in the body (every 6 hours) is -47% times the amount present and then <u>plus</u> 200 milligrams, which can be written as

$$\Delta x = -47\% \cdot x_n + 200$$
$$x_{n+1} - x_n = -0.47 \cdot x_n + 200$$
$$x_{n+1} = x_n - 0.47 \cdot x_n + 200$$
$$x_{n+1} = 0.53 \cdot x_n + 200$$

which is a linear first order difference equation where $a = 0.53$ and $b = 200$. So the equilibrium for this equation is $A = \dfrac{b}{1-a} = \dfrac{200}{1-0.53} \approx 426$.

For part (a), the maximum drug level in the body is the equilibrium, 426 milligrams.

For part (b), if $x_0 = 200$ corresponds to the first dose, then $x_1 = 306$ is the amount of the drug in the body 6 hours after the first dose, $x_2 = 362$ is the amount of the drug in the body 12 hours after the first dose, $x_3 = 392$ is the amount of the drug in the body 18 hours after the first dose, and $x_4 = 408$ is the amount of the drug in the body 24 hours after the first dose. **(Please note that the answer given for part (b) in the back of the textbook is incorrect!)**

For part (c), we want an equilibrium value of $A \le 1000$. Since $A = \dfrac{b}{1-a} = \dfrac{200}{1-a}$, this gives us a corresponding value for $a = 1 - \dfrac{b}{A} = 1 - \dfrac{200}{1000} = 0.8$. $a = 0.8$ means that 80% of the amount of the drug remains in the body or, in other words, 20% of the amount of the drug has been eliminated.

So, we need to calculate how long it will take for 20% of the amount of the drug in the body to be eliminated, and that will tell us how often we should administer the dose. 80% of 200 milligrams is 160 milligrams, so we need to solve the exponential decay equation for the amount of time it takes to decay from 200 to 160:

$$N(t) = N_0 \cdot R^t$$

$$160 = 200 \cdot (0.9)^t$$

$$0.8 = (0.9)^t$$

$$\ln(0.8) = \ln(0.9)^t$$

$$t \cdot \ln(0.9) = \ln(0.8)$$

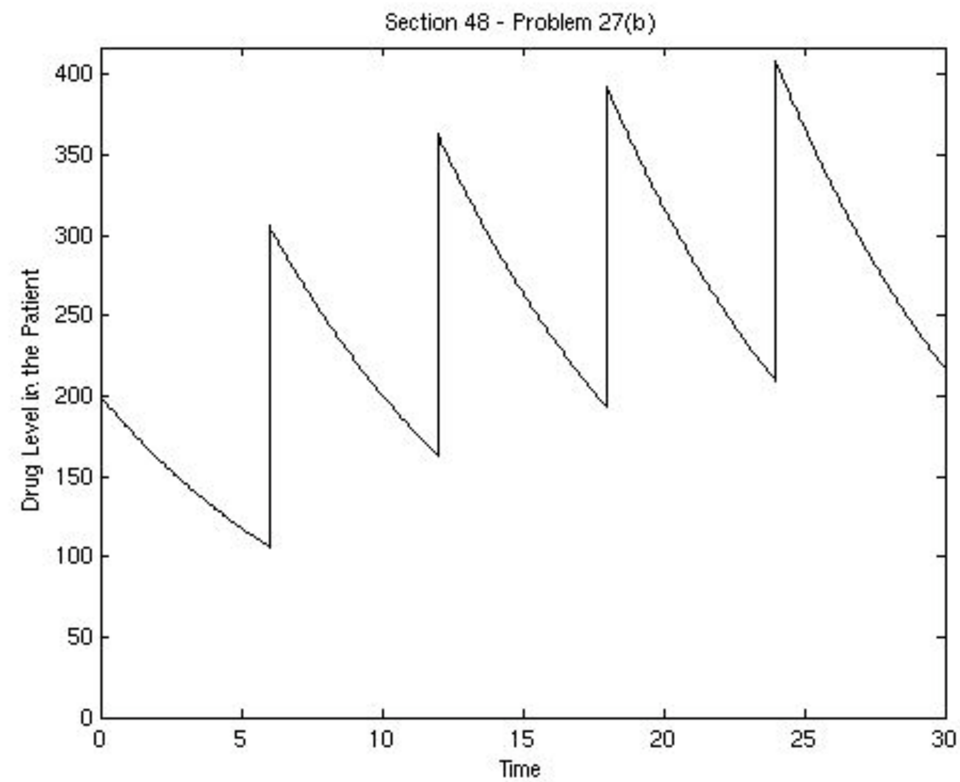
$$t = \frac{\ln(0.8)}{\ln(0.9)} \approx 2.12$$

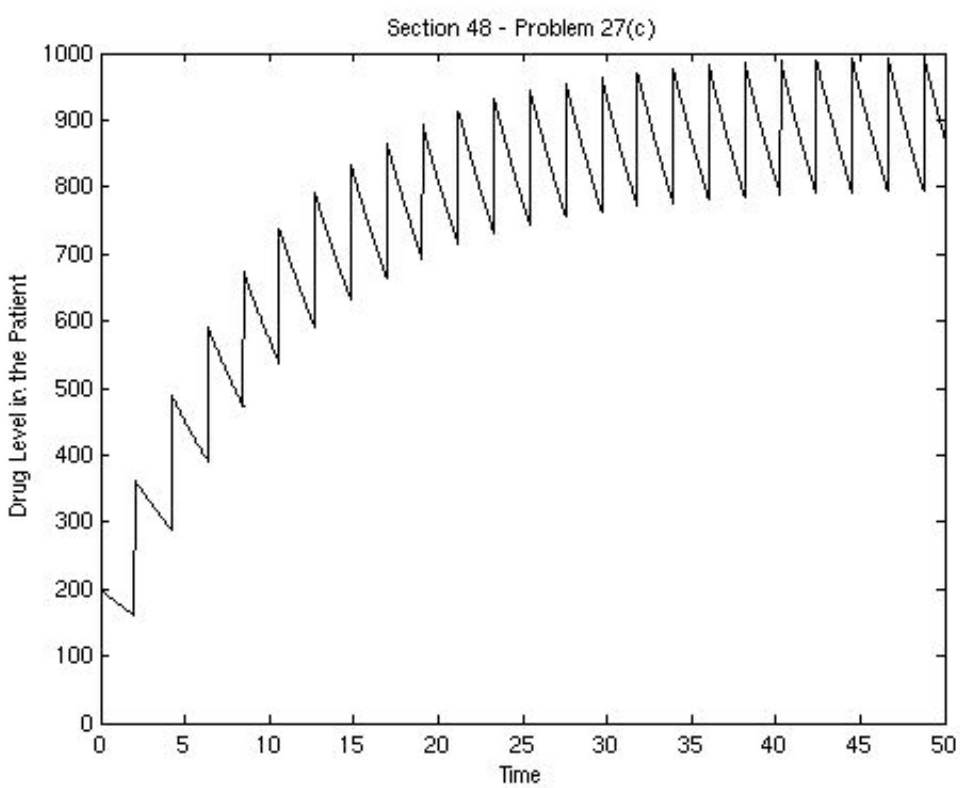
which means that it takes about 2.12 hours for the body to eliminate 20% of the amount of the drug, and that is how often we should administer the drug if we want to build up a maximum drug level of 1000 milligrams.

81% of the drug remains in the body after 2 hours (since $(0.9)^2 = 0.81$), which gives a difference equation of

$$x_{n+1} = 0.81 \cdot x_n + 200$$

and an equilibrium value of $A = \dfrac{b}{1-a} = \dfrac{200}{1-0.81} \approx 1053$, which is greater than the desired maximum of 1000. This means that we should not round the above answer down to 2 hours because administering the drug every 2 hours would result in a maximum drug level greater than 1000 milligrams.


Section 48 - Problem 27(b)
Drug Level in the Patient
Time
0
50
100
150
200
250
300
350
400
0
5
10
15
20
25
30


Section 48 - Problem 27(c)
Drug Level in the Patient
Time
0
100
200
300
400
500
600
700
800
900
1000
0
5
10
15
20
25
30
35
40
45
50

28. A level of more than 1500 milligrams of a certain drug in the body is considered unsafe. Individuals doses are 250 milligrams, and the drug is removed from the body according to the exponential decay equation

$$N(t) = N_0 e^{-0.1t}$$

where t is measured in hours. How frequently can the drug be safely administered?

If you haven't already, read through problem 27 first.

The difference equation for this problem is

$$x_{n+1} = a \cdot x_n + 250$$

where a is the proportion of drug that is left in the body after it decays exponentially between doses and $b = 250$ is the amount of each dose.

In order for the amount of the drug in the body to remain below 1500 milligrams, we want an equilibrium value of $A \le 1500$. Since $A = \dfrac{b}{1-a} = \dfrac{250}{1-a}$, this gives us a corresponding maximum value for $a = 1 - \dfrac{b}{A} = 1 - \dfrac{250}{1500} = \dfrac{5}{6}$. $a = \dfrac{5}{6}$ means that $\dfrac{5}{6}$ of the amount of the drug remains in the body or, in other words, $\frac{1}{6}$ of the amount of the drug has been eliminated.

So, we need to calculate how long it will take for $\frac{1}{6}$ of the amount of the drug in the body to be eliminated, and that will tell us how often we should administer the dose. $\frac{5}{6}$ of 250 milligrams is $208\frac{1}{3}$ milligrams, so we need to solve the exponential decay equation for the amount of time it takes to decay from 250 to $208\frac{1}{3}$:

$$N(t) = N_0 e^{-0.1t}$$

$$208\tfrac{1}{3} = 250 \cdot e^{-0.1t}$$

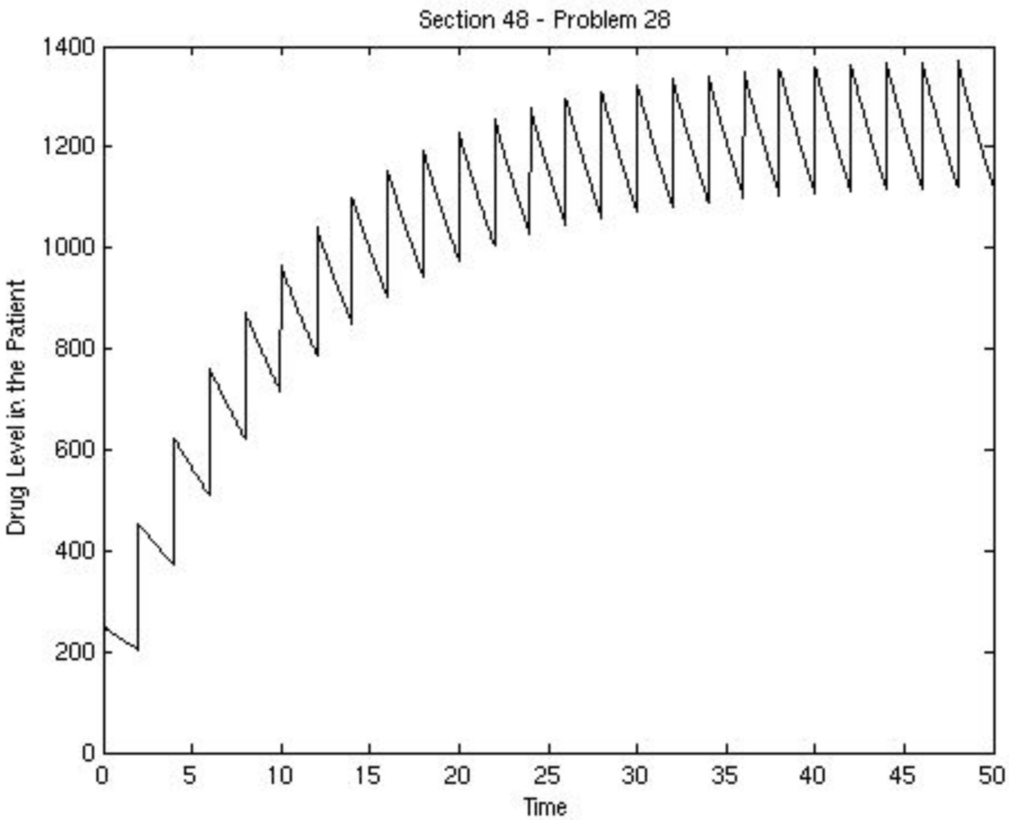
$$\tfrac{5}{6} = e^{-0.1t}$$

$$\ln\left(\tfrac{5}{6}\right) = \ln\left(e^{-0.1t}\right)$$

$$-0.1t = \ln\left(\tfrac{5}{6}\right)$$

$$t = \frac{\ln\left(\frac{5}{6}\right)}{-0.1} \approx 1.823$$

To be safe, we will round up to 2 hours. 2 hours gives a corresponding value of $a = e^{(-0.1 \cdot 2)} = 0.81873$ and an equilibrium value of $A = \dfrac{b}{1-a} = \dfrac{250}{1-0.81873} \approx 1379$, which is within the desired maximum.


Section 48 - Problem 28
Drug Level in the Patient
Time
0
200
400
600
800
1000
1200
1400
0
5
10
15
20
25
30
35
40
45
50

30. Approximately 100 new cases of a rare disease arise each year. Through drug therapy, about 25% of all individuals with the affliction are cured each year. Let x_n denote the number with the disease after n years.
(a) Find and solve the difference equation relating x_{n+1} to x_n given that $x_0 = 700$.
(b) Compute $\lim_{n\to\infty} x_n$.

The change in the number of cases each year is -25% times the current number of cases and then plus 100 new cases, which can be written as

$$\Delta x = -25\% \cdot x_n + 100$$
$$x_{n+1} - x_n = -0.25 \cdot x_n + 100$$
$$x_{n+1} = x_n - 0.25 \cdot x_n + 100$$
$$x_{n+1} = 0.75 \cdot x_n + 100$$

which is a linear first order difference equation where $a = 0.75$ and $b = 100$. So the equilibrium for this equation is $A = \dfrac{b}{1-a} = \dfrac{100}{1-0.75} = 400$, and since 0.75 is between -1 and 1, this equilibrium is stable.

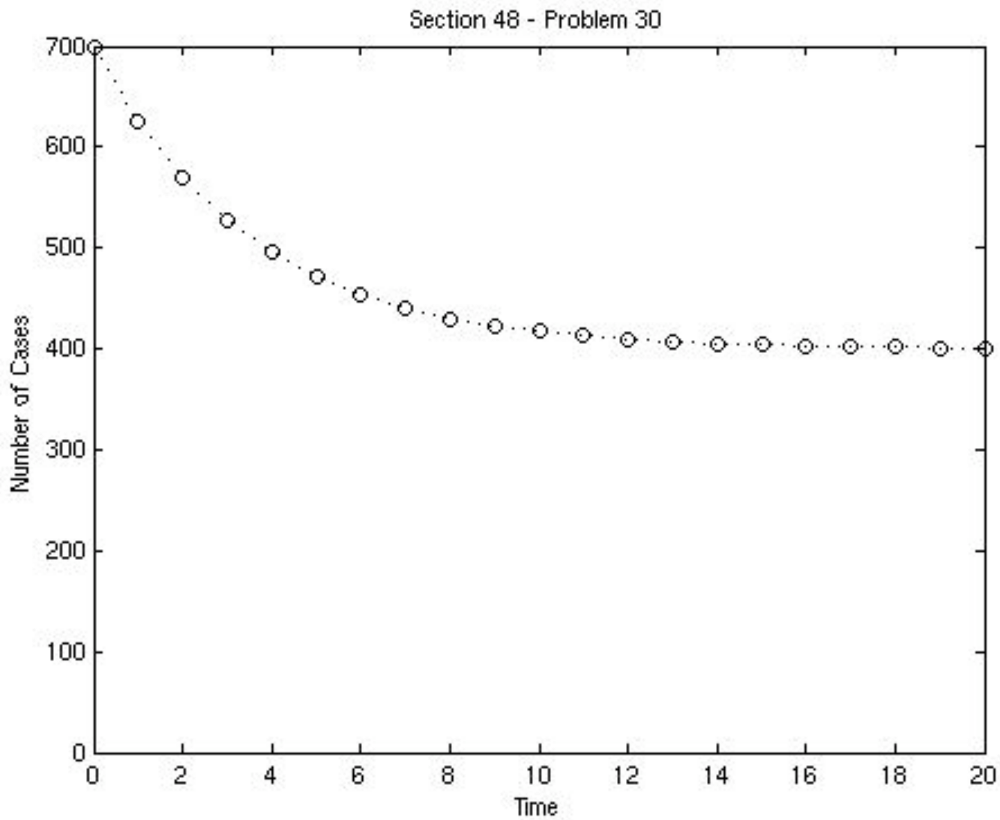
For part (a), if $x_0 = 700$, then the solution to the difference equation is

$$x_n = (x_0 - A)\cdot a^n + A$$
$$x_n = (700 - 400)\cdot (0.75)^n + 400$$
$$x_n = 300 \cdot (0.75)^n + 400$$

For part (b), since the equilibrium is stable, then $\lim_{n\to\infty} x_n = 400$. Also,

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty}\left[300 \cdot (0.75)^n + 400\right]$$
$$\lim_{n\to\infty} x_n = 300 \cdot \lim_{n\to\infty}(0.75)^n + 400$$
$$\lim_{n\to\infty} x_n = 300 \cdot 0 + 400 = 400$$

This means that in the long run, we expect that there will always be about 400 cases of the disease. Also, this makes sense because if there are 400 cases and 25% of them are cured each year, that means that 100 cases are cured each year. And then if there are 100 new cases each year, that brings the number of cases back to 400.


Section 48 - Problem 30
Number of Cases
Time
0
100
200
300
400
500
600
700
0
2
4
6
8
10
12
14
16
18
20